BC Key 2017

Review Derivatives, Differentiability and Continuity See scratch paper for work at end

Find the derivatives of the following

1. $f(x) = x^2 + 3x - 2$ $\boxed{2x+3}$

2. $f(x) = \sqrt[3]{x} + \frac{1}{x}$ $\quad x^{1/3} + x^{-1}$ $\boxed{\frac{1}{3}x^{-2/3} - x^{-2}}$

3. $y = \frac{x^2 - 4x}{\sqrt{x}}$ $\quad x^{3/2} - 4x^{1/2}$ $\boxed{\frac{3}{2}x^{1/2} - 2x^{-1/2}}$

4. $h(x) = \frac{x^2}{x-1}$ $\quad \frac{(x-1)(2x) - x^2(1)}{(x-1)^2}$ $\boxed{\frac{x^2-2x}{(x-1)^2}}$

5. $f(x) = x^2(\sqrt{x} + 1)$ $\boxed{\frac{5}{2}x^{3/2} + 2x}$

$2x \quad \frac{1}{2}x^{-1/2}$

$x^2(\frac{1}{2}x^{-1/2}) + 2x(x^{1/2}+1) = \frac{1}{2}x^{3/2} + 2x^{3/2} + 2x$

6. $y = \frac{x^2}{2}$ $\boxed{x}$

7. $h(x) = \frac{x+2}{x-1}$ $\boxed{\frac{-3}{(x-1)^2}}$ $\leftarrow \frac{(x-1)(1) - (x+2)(1)}{(x-1)^2}$

8. $g(x) = \frac{(x+1)(x+2)}{(x-1)(x-2)}$ $\boxed{\frac{-6x^2+12}{(x-1)^2(x-2)^2}}$

9. $y = \csc(x)\sec(x)$ $\boxed{\sec^2 x - \csc^2 x}$

10. $f(x) = \frac{\sqrt{x}+2}{\sqrt{x}-2}$ $\boxed{\frac{-2}{\sqrt{x}(\sqrt{x}-2)^2}}$

11. Find the equation of the tangent line to the graph of $f(x) = \frac{1}{x-1}$ $(x-1)^{-1}$ at the point where x = 2. $f(2) = 1$

$f'(x) = -1(x-1)^{-2} = \frac{-1}{(x-1)^2}$ $\quad f'(2) = \frac{-1}{1} = -1$ $\quad \boxed{y - 1 = -1(x-2)}$

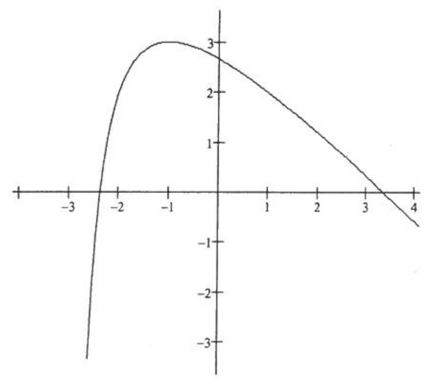
12. Find the derivative of $f(x) = 1 - x^2$ using the **definition of the derivative.**

see scratch paper $f'(x) = -2x$

13. Determine the instantaneous rate of change of the function $f(x) = x^3 - x$ at the point (2, 6).

$f'(x) = 3x^2 - 1$ $\quad f'(2) = 3(4) - 1 = 11$

Given the graph of f to the right, insert < or > between the expressions.



14. $f(0)$ ___<___ $f(-1)$.

15. $f'(-2)$ ___>___ $f'(1)$ $\quad$ + $\quad$ -

16. $f'(0)$ ___<___ $f(2)$ $\quad$ - $\quad$ +

17. The volume of a balloon changes with the radius. $V = \frac{4}{3}\pi r^3$. At what rate is the volume changing when the radius is 4cm. $\quad V' = 4\pi r^2$ $\quad V' = 4\pi(16) = \boxed{64\pi \text{ cm}^3/\text{cm}}$

18. Find the equations of the tangent line(s) to $f(x) = x^3 + 3x^2 + 2x$ which are parallel to $y = 2x + 7$

see scratch $\quad y = 2x \quad y = 2x + 4$

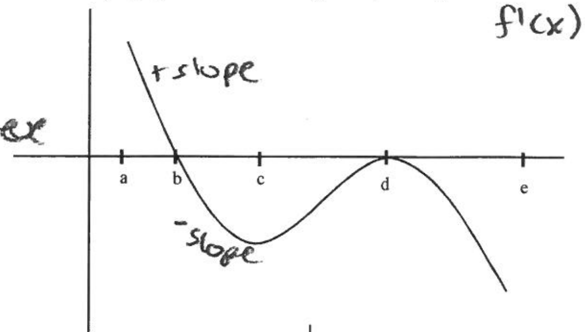
19. Mr. Bean was wondering if $h(x) = x^3 + x$ had any points where the tangents to $h(x)$ where horizontal. He took the derivative of $h(x)$, $h'(x) = 3x^2 + 1$ and set this equal to zero. When he did this he realized that the solutions to the equation were imaginary. What, if anything, can you say about $h(x)$ in this situation? no horizontal tangents.

20. Suppose the curve below is the graph of $f'(x)$, the derivative of $f(x)$. It is **NOT** the graph of $f(x)$. Determine at which of the x-values and/or intervals $f(x)$ is increasing. Explain your answer.

f increases when $f' > 0$

(a, b) because

f' graph is above the x axis



21. Determine the x-values at which $h(x)$ is not continuous.

$x = -2$

$\lim_{x \to -2^-} h(x) = 2$

$\lim_{x \to -2^+} h(x) = 0$

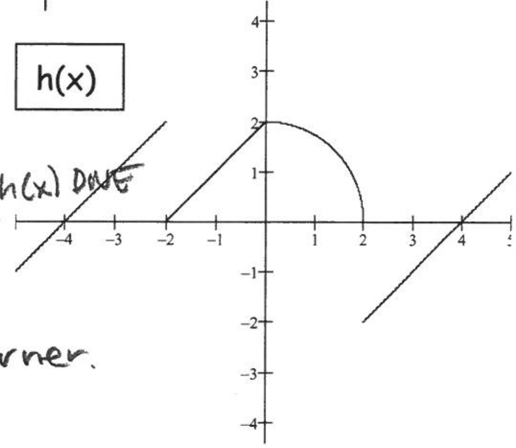
$\lim_{x \to -2} h(x)$ DNE

$x = 2$

$\lim_{x \to 2^-} h(x) = 0$

$\lim_{x \to 2^+} h(x) = -2$

$\therefore \lim_{x \to 2} h(x)$ DNE



22. Determine the x-values at which $h(x)$ is not differentiable.

$x = 2, -2$ not cont. see above

$x = 0$

$\lim_{x \to 0^-} h'(x) \neq \lim_{x \to 0^+} h'(x)$ corner.

23. Given:

g(3)	g'(3)	h(3)	h'(3)
4	-2	3	π

Find f'(3) for the following:

a. $f(x) = 4g(x) - \frac{1}{2}h(x) + 1$

$f'(x) = 4g'(x) - \frac{1}{2}h'(x)$

$f'(3) = 4(-2) - \frac{1}{2}(\pi) = \boxed{-8 - \frac{\pi}{2}}$

b. $f(x) = g(x)h(x)$

$f' = gh' + hg'$

$f'(3) = 4\pi + 3 \cdot -2$

$f'(3) = \boxed{4\pi - 6}$

c. $f(x) = \dfrac{g(x)}{2h(x)}$

$f' = \dfrac{2hg' - g \cdot 2h'}{4h^2}$

$f'(3) = \dfrac{2(3)(-2) - 4(2)(\pi)}{4(9)} = \boxed{\dfrac{-3 - 2\pi}{9}}$

Trig Differentiate

1. $f(t) = t^2 \sin t$

$\boxed{f'(t) = t^2 \cos t + 2t \sin t}$

2. $f(\theta) = (\theta + 1)\cos\theta$

$f' = (\theta + 1)(-\sin\theta) + \cos\theta$

$f' = -\theta\sin\theta - \sin\theta + \cos\theta$

3. $f(x) = \dfrac{\cos x}{x}$

$\dfrac{x(-\sin x) - \cos x}{x^2}$

$\dfrac{-x\sin x - \cos x}{x^2}$

4. $f(x) = \dfrac{\sin x}{x}$

$\dfrac{x\cos x - \sin x}{x^2}$

5. $y = 5x \csc x$

$y' = 5\csc x(-x\cot x + 1)$

6. $y = x\sin x + \cos x$

$y' = x\cos x + \sin x + (-\sin x)$

$= x\cos x$

7. $y = x^2 \sin x + 2x \cos x$

$y' = x^2\cos x + 2x\sin x + 2x(-\sin x) + 2\cos x$

$y' = x^2\cos x + 2\cos x$

8. $h(\theta) = 5\sec\theta + \tan\theta$

$h'(\theta) = 5\sec\theta\tan\theta + \sec^2\theta$

Higher level derivatives: Find f''(x) for

see scratch paper for work

1. $f(x) = \dfrac{x}{x+1}$

$f''(x) = \dfrac{-2}{(x+1)^3}$

2. $f(x) = 4\sqrt{x} - \dfrac{2}{\sqrt{x}}$

$f''(x) = \boxed{-x^{-3/2} - 3/2\, x^{-5/2}}$

$f' = 3ax^2 - 6$ $\quad 2bx$

Given: f(x) is differentiable. $f(x)=\begin{cases} ax^3-6x & x\le 1 \\ bx^2+4 & x>1 \end{cases}$ *Find a and b* $\quad x=1 \quad 3a-6=2b$

$x=1 \quad a\cdot 1-6 = b(1)+4$

$a-6=b+4$

$a=b+10$

$3(b+10)-6=2b$

$3b+30-6=2b$

$24=-b$

$\boxed{-24=b \quad a=-14}$

Chain rule practice See scratch paper

a. $y=\sec(5x)-\csc(5x)$ $\quad y' = 5\sec 5x \tan 5x + 5\csc 5x \cot 5x$

b. $f(x)=\dfrac{\sqrt{5+x^2}}{x^4+1}$ $\quad f'(x) = \dfrac{(5+x^2)^{-1/2}(x)(-3x^4-20x^2+1)}{(x^4+1)^2}$

c. $f(x)=\sin^2(5x) = (\sin 5x)^2$ $\quad f'(x) = 2\sin 5x \cdot \cos 5x \cdot 5$

$u^2 \quad 2uu'$ $\quad = 10\sin(5x)\cos(5x)$

$u=\sin 5x$

$u' = \cos(5x)\cdot 5$

d. $h(x)=\dfrac{5}{(5x^2+3)} = 5(5x^2+3)^{-1}$ $\quad h'(x) = -5(5x^2+3)^{-2}\cdot 10x = \dfrac{-50x}{(5x^2+3)^2}$

$u=5x^2+3 \quad u'=10x$

e. $y=\sin\sqrt[3]{x}$ $\quad \sin u$ $\quad \boxed{y' = \cos(x^{1/3})\cdot \tfrac{1}{3}x^{-2/3}}$

$\sin x^{1/3}$

$u=x^{1/3} \quad u' = \tfrac{1}{3}x^{-2/3}$

Motion and other problems.

1. A particle moves along the x -axis so that it's position is given by $x(t)=t^3-6t^2+9t+11$

where x(t) is measured in inches and t is in seconds, t≥0. Justify all answers. Use appropriate units.

a. Find the velocity and acceleration functions. $x'(t) = v(t) = 3t^2-12t+9 = 3(t^2-4t+3)$

$x''(t) = a(t) = 6t-12$

$t^2-4t+3=0$

$(t-3)(t-1)=0$

$t=1$

$t=3$

b. When is the particle travelling left? $v<0 \quad (1,3)$

c. When is the particle stopped? $t=1$ sec $\quad t=3$ sec $\quad v=0$ because

d. What is the average velocity from [0,3] $\dfrac{s(3)-s(0)}{3} = \dfrac{11-11}{3} = 0$ in/s

v: + (1) − (3) +

e. What is the acceleration at t=1? $a(1)=6(1)-12 = -6$ in/s²

− (2) +

f. What is the velocity when the acceleration is 0? $v(2)=3(4)-12(4)+9 = -3$ in/sec

$0=6t-12 \quad t=2$

g. When is the particle speeding up? Slowing down? See scratch

2. **Yes calculator**. For [0,6] seconds, a particle moves along the x axis. The particle's position function is unknown. The velocity is given by $v(t)=2\sin(e^{\frac{t}{4}})+1$. v is ft/sec. radian mode!

a. When is the particle moving left?

b. Is the speed of the particle increasing or decreasing at t=5.5?

c. Find the average acceleration from [0,6].

☆ make sure you are in radians

2. **Yes calculator**. For [0,6] seconds, a particle moves along the x axis. The particle's position function is unknown. The velocity is given by $v(t) = 2\sin(e^{\frac{t}{4}}) + 1$. V = ft/sec

x intercept: 5.196

a. When is the particle moving left?

(5.196, 6) V<0

b. Is the speed of the particle increasing or decreasing at t=5.5?

increasing V<0 a<0

$\frac{\Delta V}{\Delta t}$ c. Find the average acceleration from [0,6]. $\frac{V(6)-V(0)}{6-0} = \frac{-.947-2.683}{6} = \frac{-3.63}{6} = -0.60$ ft/

3. The data in the table below give selected values for the velocity, in meters/minute, of a particle moving along the x-axis. The velocity v is a differentiable function of time t.

Time t (min)	0	2	5	6	8	12
Velocity $v(t)$ (meters/min)	-3	2	3	5	7	5

a. At $t = 0$, is the particle moving to the right or to the left? Explain your answer.

left because v(0) < 0

b. Is there a time during $0 \le t \le 12$ minutes when the particle is at rest? Explain.

yes (0,2) since v(t) is differentiable, by the IVT v(t) = 0 between (0,2).

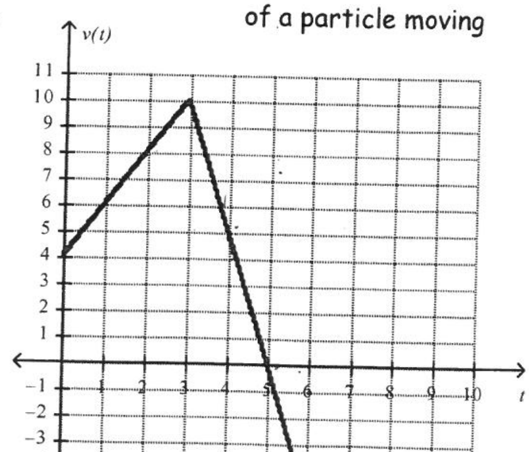
c. Use data from the table to find an approximation for $v'(10)$ and explain the meaning of $v'(10)$ in terms of the motion of the particle. Show the computation that lead to your answer and indicate units of measure. use t = 8 t = 12

$\frac{5-7 \text{ m/min}}{12-8 \text{ min}} = \frac{-2}{4} = -\frac{1}{2} \frac{m}{min^2}$ the acceleration is approx. $-\frac{1}{2}$ m/min² at t = 10

since V>0, the particle is slowing down since A<0

d. Is the particle speeding up or slowing down at t=10. (use part c to help).

4. The graph represents the velocity $v(t)$, in feet per second, of a particle moving along the x-axis over the time interval $0 \le t \le 9$ seconds.



a. At $t = 4$ seconds, is the particle moving to the right or the left? Explain your answer.

right V(4) > 0

b. Over what time interval is the particle moving to the left? Explain your answer. (5, 9)

V(t) < 0

c. At $t = 4$ seconds, is the acceleration of the particle positive or negative? Explain your answer. negative V' < 0

d. What is the average acceleration of the particle over the interval $2 \le t \le 4$? Show the computations that lead to your answer and indicate units or measure.

(2, 8) (4, 5)

$\frac{5-8}{4-2} = \boxed{-\frac{3}{2} \text{ ft/sec}^2}$

☆ make sure you are in radians

2. **Yes calculator**. For [0,6] seconds, a particle moves along the x axis. The particle's position function is unknown. The velocity is given by $v(t) = 2\sin(e^{\frac{t}{4}}) + 1$. V = ft/sec

x intercept: 5.196

a. When is the particle moving left?

(5.196, 6) v<0

b. Is the speed of the particle increasing or decreasing at t=5.5?

increasing V<0 a<0

$\frac{\Delta V}{\Delta t}$ c. Find the average acceleration from [0,6]. $\frac{V(6)-V(0)}{6-0} = \frac{-.947-2.683}{6} = \frac{-3.63}{6} = -0.60$ ft,

3. The data in the table below give selected values for the velocity, in meters/minute, of a particle moving along the *x*-axis. The velocity *v* is a differentiable function of time *t*.

Time *t* (min)	0	2	5	6	8	12
Velocity $v(t)$ (meters/min)	-3	2	3	5	7	5

a. At $t = 0$, is the particle moving to the right or to the left? Explain your answer.

left because v(0) < 0

b. Is there a time during $0 \le t \le 12$ minutes when the particle is at rest? Explain.

yes (0,2) Since V(t) is differentiable, by the IVT V(t)=0 between (0,2).

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d. Is the particle speeding up or slowing down at t=10. (use part c to help). since V>0, the particle is slowing down since A<0

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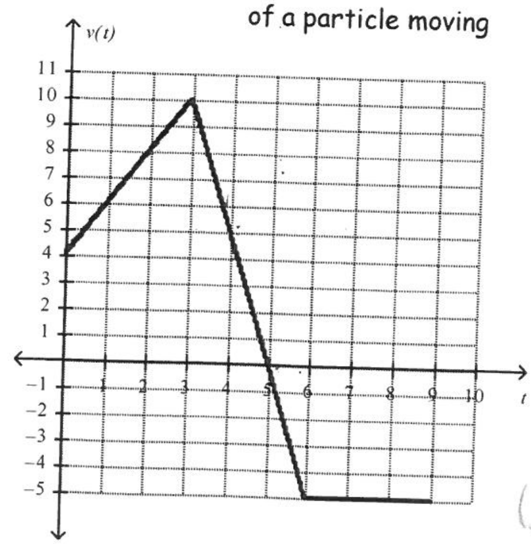


a. At $t = 4$ seconds, is the particle moving to the right or the left? Explain your answer.

right V(4) > 0

b. Over what time interval is the particle moving to the left? Explain your answer. (5,9) V(t)<0

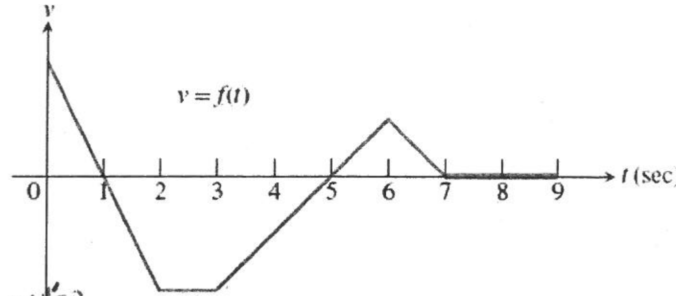
c. At $t = 4$ seconds, is the acceleration of the particle positive or negative? Explain your answer. negative V' < 0

d. What is the average acceleration of the particle over the interval $2 \le t \le 4$? Show the computations that lead to your answer and indicate units or measure.

(2,8) (4,5)

$\frac{5-8}{4-2} = \boxed{-\frac{3}{2} \text{ ft/sec}^2}$

5. The accompanying figure shows the velocity $v = f(t)$ of a particle moving on a coordinate line.



When does the particle:

(a) move forward? $V>0$ (0,1) (5,7)

(b) Move backward? $V<0$ (1,5)

(c) Speed up? (1,2) (5,6)

(d) Slow down? (0,1) (3,5) (6,7)

When is the particle acceleration

(e) positive? $v'>0$ (3,6)

(f) Negative? $v'<0$ (0,2) (6,7)

(g) Zero? $v'=0$ (2,3) (7,9)

(h) When does the particle move at its greatest speed? (2,3) greatest abs. value of v

(i) When does the particle stand still for more than an instant? (7,9) $v=0$

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6. A particle P moves on the number line. The figure shows the position of P as a function of time(t).

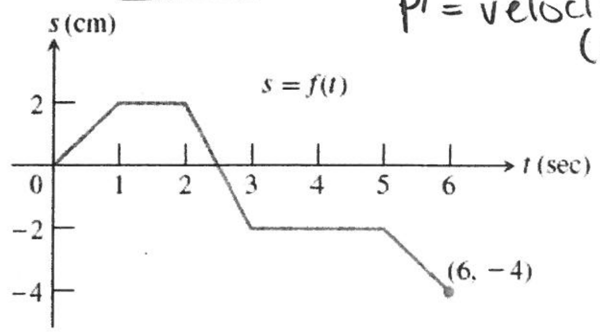
$P' =$ velocity (slope)

a. When is P moving to the left? $P'<0$ (2,3) (5,6)

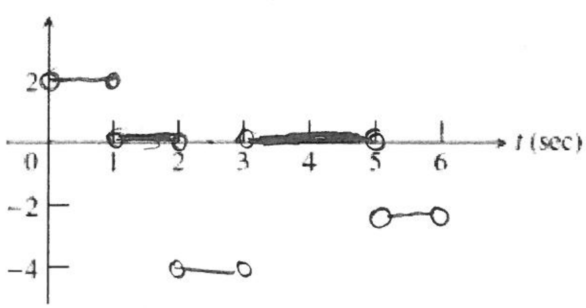
b. Moving to the right? $P'>0$ (0,1)

c. Standing still? (1,2) (3,5) $P'=0$

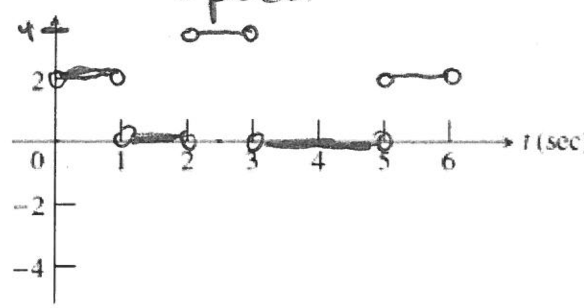
Graph the particle's velocity and speed.



Velocity



Speed



$|v|$

Basic Derivative Review

1) $f(x) = x^2 + 3x - 2$

$f'(x) = \boxed{2x+3}$

2) $f(x) = \sqrt[3]{x} + 1/x = x^{1/3} + x^{-1}$

$f'(x) = \boxed{1/3\, x^{-2/3} - x^{-2}}$

or $\frac{1}{3\sqrt[3]{x^2}} - \frac{1}{x^2}$

3) $y = \frac{x^2 - 4x}{\sqrt{x}} = \frac{x^2}{x^{1/2}} - \frac{4x}{x^{1/2}}$

$= x^{3/2} - 4x^{1/2}$

$\boxed{y' = \frac{3}{2}x^{1/2} - 2x^{-1/2}}$

4) $h(x) = \frac{x^2}{x-1}$

$h'(x) = \frac{(x-1)(2x) - x^2(1)}{(x-1)^2}$

$h'(x) = \boxed{\frac{x^2-2x}{(x-1)^2}}$

5) $f(x) = x^2(\sqrt{x} + 1)$

$2x \quad \frac{1}{2}x^{-1/2}$

$f'(x) = x^2(\frac{1}{2}x^{-1/2}) + (x^{1/2}+1)2x$

$f'(x) = \frac{1}{2}x^{3/2} + 2x^{3/2} + 2x$

$\boxed{f'(x) = \frac{5}{2}x^{3/2} + 2x}$

6. $y = 1/2\, x^2$

$\boxed{y' = x}$

7. $h(x) = \frac{x+2}{x-1}$

$h'(x) = \frac{(x-1)(1) - (x+2)(1)}{(x-1)^2}$

$\boxed{h'(x) = -3/(x-1)^2}$

8. $g(x) = \frac{x^2+3x+2}{x^2-3x+2}$ $\quad 2x+3 \quad 2x-3$

$g'(x) = \frac{(x^2-3x+2)(2x+3) - (x^2+3x+2)(2x-3)}{(x-1)^2(x-2)^2}$

$2x^3 + 3x^2 - 6x^2 - 9x + 4x + 6$

$-(2x^3 - 3x^2 + 6x^2 - 9x + 4x - 6)$

$-6x^2 + 12$

$g'(x) = \frac{-6x^2+12}{(x-1)^2(x-2)^2}$

9) $y = \csc x \sec x$

$\csc x$	$\sec x$
$-\csc x \cot x$	$\sec x \tan x$

$y' = \csc x(\sec x \tan x) + \sec x(-\csc x \cot$

$\frac{1}{\cancel{\sin x}} \frac{1}{\cos x} \frac{\cancel{\sin x}}{\cos x} - \frac{1}{\cos x} \frac{1}{\sin x} \frac{\cos}{\sin}$

$\frac{1}{\cos^2 x} - \frac{1}{\sin^2 x}$

$\boxed{\sec^2 x - \csc^2 x}$

10. $f(x) = \frac{\sqrt{x}+2}{\sqrt{x}-2}$ $\quad \frac{1}{2}x^{-1/2}$ $\quad \frac{1}{2}x^{-1/2}$

$f'(x) = \frac{(x^{1/2}-2)(\frac{1}{2}x^{-1/2}) - (x^{1/2}+2)(\frac{1}{2}x^{-1/2})}{(x^{1/2}-2)^2}$

$\frac{1}{2} - x^{-1/2} - \frac{1}{2} - x^{-1/2}$

$\frac{-2x^{-1/2}}{(x^{1/2}-2)^2} \quad \frac{-2}{\sqrt{x}(\sqrt{x}-2)^2}$

12. $\lim_{h \to 0} \frac{(1-(x+h)^2) - (1-x^2)}{h}$

$\lim_{h \to 0} \frac{1 - x^2 - 2xh - h^2 - 1 + x^2}{h} = \lim_{h \to 0} \frac{-2xh - h^2}{h}$

$= \lim_{h \to 0} \frac{\cancel{h}(-2x-h)}{\cancel{h}} = \boxed{-2x}$

18. $f'(x) = 3x^2 + 6x + 2$ $\qquad // \; y = 2x+7 \qquad m = 2$

$2 = 3x^2 + 6x + 2$

$0 = 3x(x+2)$

$x = 0 \quad x = -2$

$f(0) = 0 \quad f(-2) = 0$

$y - 0 = 2(x-0)$ $\qquad$ $y - 0 \;\; 2(x+2)$

$\boxed{y = 2x}$ $\qquad$ $\boxed{y = 2x+4}$

Higher Level Deriv.

1. $f(x) = \dfrac{x}{x+1}$

$f'(x) = \dfrac{(x+1)(1) - x(1)}{(x+1)^2}$

$f'(x) = \dfrac{1}{(x+1)^2} = (x+1)^{-2}$

$f''(x) = -2(x+1)^{-3}(1)$
$= \dfrac{-2}{(x+1)^3}$

$u = x+1 \quad u' = 1$

2. $4x^{1/2} - 2x^{-1/2}$

$f' = 2x^{-1/2} + x^{-3/2}$

$f''(x) = -x^{-3/2} - \frac{3}{2}x^{-5/2}$

Chain rule Practice

a) $y = \sec 5x - \csc 5x$ $\qquad u = 5x \quad u' = 5$

$y' = 5\sec 5x \tan 5x + 5\csc 5x \cot 5x$

b) $f(x) = \dfrac{\sqrt{5+x^2}}{x^4+1}$ $\qquad u = 5+x^2 \quad u' = 2x \qquad u^{1/2} \quad \frac{1}{2}u^{-1/2} \cdot u'$

$\longrightarrow \frac{1}{2}(5+x^2)^{-1/2} \cdot 2x = x(5+x^2)^{-1/2}$

$f' = \dfrac{(x^4+1)(x)(5+x^2)^{-1/2} - (5+x^2)^{1/2}(4x^3)}{(x^4+1)^2}$

$(5+x^2)^{-1/2}(x)\,[x^4+1 - (5+x^2)(4x^2)]$

$[x^4+1 - 20x^2 - 4x^4]$

$(-3x^4 - 20x^2 + 1)$

$f'(x) = \dfrac{(5+x^2)^{-1/2}(x)(-3x^4 - 20x^2 + 1)}{(x^4+1)^2}$

motion

1.9

V: + | 1 | − | 3 | +

A: − | 2 | +

A: − | 1 | − | 2 | + | 3 | +

V: + | 1 | − | 2 | − | 3 | +

Slow Down: $(0,1) \cup (2,3)$ V & A opposite signs

speed up $(1,2)$ a & v both −

$(3,\infty)$ a & v both +